



# CONTINUATION FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS DISCRETIZED BY THE MULTIQUADRIC METHOD

A. I. FEDOSEYEV\*

*Center for Microgravity and Materials Research,  
University of Alabama in Huntsville,  
Huntsville, AL 35899, USA*

M. J. FRIEDMAN†

*Department of Mathematical Sciences,  
University of Alabama in Huntsville,  
Huntsville, AL 35899, USA*

E. J. KANSA‡

*Lawrence Livermore National Laboratory,  
Livermore, CA 94551, USA*

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The Multiquadric Radial Basis Function (MQ) Method is a meshless collocation method with global basis functions. It is known to have exponential convergence for interpolation problems. We discretize nonlinear elliptic PDEs by the MQ method. This results in modest-size systems of nonlinear algebraic equations which can be efficiently continued by standard continuation software such as AUTO and CONTENT. Examples are given of detection of bifurcations in 1D and 2D PDEs. These examples show high accuracy with small number of unknowns, as compared with known results from the literature.

## 1. Introduction

Nonlinear multidimensional elliptic partial differential equations (PDEs) are the basis for many scientific and engineering problems, such as viscous fluid flow phenomena, chemical reactions, crystal growth processes, pattern formation in biology, etc. In these problems it is crucial to understand the qualitative dependence of the solution on the problem parameters.

During the past two decades the numerical continuation approach has become popular for qualitative study of solutions to nonlinear equations,

see e.g. [Rheinboldt, 1986; Allgower & Georg, 1990; Doedel *et al.*, 1991; Seydel, 1998] and references therein. Several software packages, such as AUTO97 [Doedel *et al.*, 1997] and CONTENT [Kuznetsov & Levitin, 1995–1997], are currently available for bifurcation analysis of systems of nonlinear algebraic equations and ODEs, with only limited bifurcation analysis for 1D elliptic PDEs. For 2D PDEs, we mention the software package PLTMG [Bank, 1998] that allows to solve a class of boundary value problems on regions in the plane, to continue the solution with respect to a parameter and even to compute limit and branching

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\*E-mail: alex@uahtitan.uah.edu

†E-mail: friedman@math.uah.edu

‡E-mail: kansa1@llnl.gov

points. This software combines a sophisticated finite element discretization with advanced linear algebra techniques. Numerical continuation for 1D and 2D elliptic PDEs is currently an active research area, see e.g. [Neubert, 1993; Shroff & Keller, 1993; Schwetlick *et al.*, 1996; Chien *et al.*, 1997; Davidson, 1997; Kuznetsov *et al.*, 1996; Chien & Chen, 1998; Doedel & Sharifi, to appear] and [Govaerts, 2000, Chap. 10] for reaction–diffusion equations; and [Mamun & Tuckermann, 1995; Poliashenko & Aidun, 1995] for CFD. The typical approaches used are based on the finite element or finite difference discretization of the PDEs. They result in very large (thousands or tens of thousands for 2D problems) systems of nonlinear algebraic equations with sparse matrices. The continuation process is typically based on the predictor–corrector algorithms that require solving nonlinear systems by the Newton type method at each continuation step. For the bifurcation analysis during the continuation process, one usually needs to compute at least a few eigenvalues of the Jacobian matrix at each continuation step. The methods currently used both for the continuation and the corresponding eigenvalue problems are variants of Krylov subspace methods and recursive projection (RPM). Solving the resulting linear system and the eigenvalue problem require sophisticated algorithms and considerable computer resources (CPU time, memory, disk space, etc.).

In this paper we report results of numerical experiments with continuation and detection of bifurcations for 1D and 2D elliptic PDEs discretized by the Multiquadric Radial Basis Function (MQ RBF or, simply, MQ) method. The MQ method, introduced for solving PDEs in [Kansa, 1990a, 1990b], is a meshless collocation method with global basis functions which leads to finite-dimensional problems with full matrices. It was shown to give very high accuracy with a relatively small number of unknowns (tens or hundreds for 2D problems). The corresponding linear systems can be efficiently solved by direct methods. This opens a possibility for using standard continuation software, such as AUTO and CONTENT, designed for bifurcation analysis of modest-size problems. We also note that the MQ method does not require predetermined location of the nodes as the spectral method does.

In Sec. 2 we summarize previous results on solving PDEs by the MQ method. In Sec. 3 we formulate an adaptation of the MQ method suitable for the discretization of parametrized elliptic PDEs. In Sec. 4 we present results of our numerical

experiments with a 1D eigenvalue problem and in Sec. 5 we present results of our numerical experiments with continuation of solutions and detection of bifurcations for 1D and 2D elliptic PDEs. In Sec. 6 we discuss our results.

## 2. Review of the MQ Method for Elliptic PDEs

The concept of solving PDEs using radial basis functions (RBFs) was introduced by Kansa in 1990 [1990a, 1990b]. He implemented this approach for the solution of hyperbolic, parabolic, and elliptic PDEs using the MQ RBFs proposed by Hardy [1971, 1990] for interpolation of scattered data.

There exists an infinite class of RBFs. An RBF is a function  $f_j(x) \in \mathbb{R}$ ,  $x \in \mathbb{R}$  (say, in 1D case) that depends only upon the distance between  $x$  and a reference node  $x_j$ . A MQ RBF is  $g_j(c_j, x) = ((x - x_j)^2 + c_j^2)^{1/2}$ , where  $c_j$  is called the *shape parameter*. In a comprehensive study, Franke [1982] compared (global) RBFs against many popular compactly supported schemes for 2D interpolation, and he found that the global RBF schemes were superior on six criteria. Among the studied RBFs still only the MQ RBFs are proven to have an exponential convergence for the function interpolation [Madych & Nelson, 1990; Wu & Shaback, 1993]. Madych [1992] showed theoretically that the MQ interpolation scheme converges faster as the constant MQ shape parameter becomes progressively larger.

The numerical experiments for parabolic and elliptic PDEs by Kansa [1990a, 1990b] and Golberg and Chen [1996] show high accuracy and efficiency of the MQ scheme. For example, for a 1D convection–diffusion problem, Kansa [1990b] showed that the MQ solution with 20 nodes had the maximum norm error within  $10^{-4}$ , while a second-order finite difference scheme with  $K = 200$  nodes and an optimal combination of the central and upwind differences for the problem resulted in a much larger error of  $3 \cdot 10^{-2}$ .

In the numerical experiments with a nonlinear time-dependent problem modeling the 1D von Neumann blast wave Kansa [1990b] compared the exact solution and its derivatives with the MQ solution (35 nodes) and with a second-order finite difference one. The error in the maximum norm for pressure, density, energy and their gradients was  $10^{-6}$  or less for the MQ method, and in the range from  $10^{-4}$  to  $10^{-2}$  for the finite difference method with 5000 nodes.

Golberg and Chen [1997] showed that the solution of the 3D Poisson equation in an ellipsoid could be obtained with only 60 randomly distributed nodes to the same degree of accuracy as a FEM solution with 71 000 linear elements.

Sharan *et al.* [1997] showed that the MQ method yields accurate solutions for 2D Poisson and biharmonic equation, and that the MQ approach is simple to implement on domains with irregular boundaries. Cook *et al.* [1993] noted many benefits of using MQ RBFs to solve an initial value problem for a 3D nonlinear equation for the collision of two black holes. The resulting discrete system had 2000 unknowns and was solved directly.

Buhmann [1995] showed that RBFs and, in particular, MQ RBFs are useful for constructing prewavelets and wavelets. Wavelets are most frequently used in time-series analysis, but there are results for using wavelets to solve PDEs [Fasshauer & Jerome, 1999; Narcowich *et al.*, 1999]. As Buhmann points out, one can generate true wavelets by an orthonormalization process. The wavelets are an elegant way to achieve the same results as multigrid schemes. The MQ RBFs are attractive for prewavelet construction due to exceptional rates of convergence and their infinite differentiability.

The paper by Franke and Schaback [1998] provides the first theoretical analysis for solving PDEs by collocation using the RBF methods.

Kansa and Hon [1998] studied several methods for solving linear systems that arise from the MQ collocation problems. They studied the 2D Poisson equation, and showed that ill-conditioning of the system could be circumvented by using block-partitioning methods.

Kansa [1990b] introduced the concept of variable shape parameters  $c_j$  in the MQ scheme that appeared to work well in some cases. In [Kansa & Hon, 1998], a recipe for selecting  $c_j$  based upon the local radius of curvature of the solution surface was found to give better results than a constant  $c_j$  MQ scheme. Kansa and Hon [1998] tested the MQ method for the 2D Poisson equation with a set of exact solutions  $F = \exp(ax + by)$ ,  $\cos(ax + by)$ ,  $\sin(ax + by)$ ,  $\log(ax + by + c)$ ,  $\exp(-a(x - 1/2)2 - b(y - 1/2)2)$  and  $\arctan(ax + by)$ . They obtained an accuracy up to  $10^{-5}$  using a modest size, 121, set of nodes, while locally adapting the shape parameter  $c_j$ .

The multizone method of Wong *et al.* [1999] is yet another alternative method for improving computational efficiency. This method is readily

parallelizable, and the conditioning of the resulting matrices are much better.

Hon and Mao [1998] showed that an adaptive algorithm that adjusted the nodes to follow the peak of the shock wave can produce accurate results in 1D Burgers equation with only 10 nodes, even for steep shocks with  $\text{Re} = 10^4$ .

### 3. Discretization of Nonlinear Elliptic PDEs by the MQ Method

We consider the second-order system of  $n$  parametrized nonlinear elliptic partial differential equations

$$D(\alpha)\Delta u - f(\nabla u, u, x, y, \alpha) = 0, \tag{1}$$

$$\alpha \in \mathbb{R}, u(\cdot), f(\cdot) \in \mathbb{R}^n, (x, y) \in \Omega \subset \mathbb{R}^2,$$

where  $D(\alpha)$  is a positive diagonal  $n \times n$  matrix that is dependent smoothly on  $\alpha$ , subject to boundary conditions

$$f^b \left( \frac{\partial u}{\partial n}, u, x, y, \alpha \right) \Big|_{\partial\Omega} = 0, \quad f^b(\cdot) \in \mathbb{R}^n. \tag{2}$$

Here  $\alpha$  is a control parameter, and we are interested in studying the dependence of the solutions to the boundary-value problem (1), (2) on  $\alpha$ .

We discretize the continuous problem by the multiquadric radial basis function (MQ) method [Kansa, 1990a, 1990b; Madych & Nelson, 1990] as follows. Introduce a set  $\Theta_h$  of nodes ( $N$  internal and  $N_b$  on the boundary)

$$\Theta_h = \{(x_i, y_i) |_{i=1, N} \subset \Omega, \\ (x_i, y_i) |_{i=N+1, N+N_b} \subset \partial\Omega\}$$

and look for the approximate solution to (1), (2) in the form [Madych & Nelson, 1990]

$$u_h(x, y) = a_0 + \sum_{j=1}^{j=N+N_b} a_j g_j(c_j, x, y), \tag{3}$$

where  $\sum_{j=1}^{j=N+N_b} a_j = 0$ . We use this relationship to eliminate  $a_N$  from (3) which results in

$$u_h(x, y) \\ = a_0 + \sum_{j=1}^{j=N-1} a_j (g_j(c_j, x, y) - g_N(c_N, x, y)) \\ + \sum_{j=N+1}^{j=N+N_b} a_j (g_j(c_j, x, y) - g_N(c_N, x, y)), \tag{4}$$

where  $a_j \in \mathbb{R}^n$  are the unknown expansion coefficients and

$$g_j(c_j, x, y) = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c_j^2},$$

$$j = 1, \dots, N + N_b,$$

are the MQ basis functions, and  $c_j > 0$  are called *shape parameters* [Kansa, 1990b]. We then substitute  $u_h(x, y)$  into (1), (2) and use collocation at the nodes  $\Theta_h$  to obtain a finite-dimensional system

$$\begin{aligned} &\varphi_i(a^1, a^2, \alpha) \\ &\equiv D(\alpha)\Delta u_h(x_i, y_i) \\ &\quad - f(\nabla u_h(x_i, y_i), u_h(x_i, y_i), x_i, y_i, \alpha) = 0, \\ &\quad i = 1, \dots, N, \end{aligned} \tag{5}$$

$$\begin{aligned} &\varphi_{i-N}^b(a^1, a^2, \alpha) \\ &\equiv f^b\left(\frac{\partial u_h(x_i, y_i)}{\partial n}, u_h(x_i, y_i), x_i, y_i, \alpha\right) = 0, \\ &\quad i = N + 1, \dots, N + N_b, \end{aligned} \tag{6}$$

where  $a^1 = (a_0, \dots, a_{N-1}) \in \mathbb{R}^{n \times N}$ ,  $a^2 = (a_{N+1}, \dots, a_{N+N_b}) \in \mathbb{R}^{n \times N_b}$ . We next modify the discretized system to make it more suitable for continuation and bifurcation analysis. (1) We eliminate  $a^2$ , associated with the boundary nodes, so as to minimize the number of unknowns. (2) We reformulate the resulting problem in terms of (internal) nodal values  $u_i \equiv u_h(x_i, y_i)$ ,  $i = 1, \dots, N$ , so as to have the correct eigenvalue problem (to avoid dealing with matrix pencils) for the Jacobian matrix of (5) for detecting bifurcations during the continuation process. This is accomplished as follows.

1. We solve (6) for  $a^2$  (assuming that the Implicit Function Theorem is applicable here) to obtain

$$\begin{aligned} &a^2 = \psi(a^1, \alpha), \text{ or, in components,} \\ &\quad a_j = \psi_j(a^1, \alpha), \\ &\quad j = N + 1, \dots, N + N_b. \end{aligned} \tag{7}$$

Substituting this into (5) and using the notation  $\varphi = (\varphi_1, \dots, \varphi_N)$ , yields

$$\varphi(a^1, \psi(a^1, \alpha), \alpha) = 0, \quad \varphi(\cdot) \in \mathbb{R}^{n \times N}. \tag{8}$$

2. We now want to reformulate (8) in terms of the nodal values  $U = (u_1, u_2, \dots, u_N) \in \mathbb{R}^{n \times N}$ . To

this end, we first eliminate  $a^2$  from (4) by substituting (7) into (4) to obtain

$$\begin{aligned} &u_h(x, y) \\ &= a_0 + \sum_{j=1}^{j=N-1} a_j(g_j(c_j, x, y) - g_N(c_N, x, y)) \\ &\quad + \sum_{j=N+1}^{j=N+N_b} \psi_j(a^1, \alpha)(g_j(c_j, x, y) \\ &\quad - g_N(c_N, x, y)). \end{aligned} \tag{9}$$

We now define the map  $\Gamma : a^1 \mapsto U = \Gamma(a^1)$ . For  $i = 1, \dots, N$ :

$$\begin{aligned} &u_i = a_0 + \sum_{j=1}^{j=N-1} (g_j(c_j, x_i, y_i) \\ &\quad - g_N(c_N, x_i, y_i))a_j \\ &\quad + \sum_{j=N+1}^{j=N+N_b} (g_j(c_j, x_i, y_i) \\ &\quad - g_N(c_N, x_i, y_i))\psi_j(a^1, \alpha) \end{aligned} \tag{10}$$

Finally, substituting  $a^1 = \Gamma^{-1}(U)$  into (8), we arrive at the finite-dimensional continuation problem

$$G(U, \alpha) = 0, \quad U, G(\cdot) \in \mathbb{R}^{n \times N}, \alpha \in \mathbb{R}, \tag{11}$$

where

$$\begin{aligned} &G(U, \alpha) = \varphi(\Gamma^{-1}(U), \psi(\Gamma^{-1}(U), \alpha), \alpha), \\ &\quad \Gamma : \mathbb{R}^N \rightarrow \mathbb{R}^N, \psi(\cdot) \in \mathbb{R}^{n \times N_b}. \end{aligned}$$

**Remark 1.** Note that in the case that the boundary condition (2) is linear,  $\psi_j$  are linear, and consequently  $\Gamma$  is an  $N \times N$  matrix.

In Sec. 5 we consider examples of continuation of 1D PDEs with  $\Omega = (0, 1)$  and 2D PDEs with  $\Omega = (0, 1) \times (0, 1)$ . In all 2D examples we have the same number of nodes  $N_s$  in  $x$  and  $y$  directions. We choose a constant shape parameter  $c_j = s/(N_s - 1)$ . Our typical choice for  $s$  is  $4 \leq s \leq 12$ .

We use two types of node distributions. In the case of uniform node distribution  $(x_k, y_l) = (kh, lh)$ ,  $k, l = 0, \dots, N_s$ ,  $h = 1/N_s$ . In the case

of nonuniform node distribution, the nodes adjacent to the boundary  $\partial\Omega$  are placed at the distance  $\tilde{h} = h_1 h$  from  $\partial\Omega$ ,  $0.1 \leq h_1 \leq 0.5$ , while the remaining nodes are distributed uniformly. A criteria for the choice of  $h_1$  was a minimum of  $L_2$ -norm of the residual in  $\Omega$ .

#### 4. Numerical Experiments for a 1D Eigenvalue Problem

Accurate approximation of eigenvalue problems is essential for bifurcation analysis of PDEs. We have not found references in literature on the MQ-solution of eigenvalue problems. We therefore present here results for an eigenvalue problem for 1D Laplace operator. For details on the MQ discretization see Sec. 3. This is a scalar problem

$$-u'' = \lambda u, \quad u(0) = u(1) = 0, \quad (12)$$

that has the exact solution:

$$(\lambda_m, U^m(x)) = ((\pi m)^2, \sin(\pi m x)), \\ m = 1, 2, \dots$$

where  $(\lambda_m, U^m(x))$  is the  $m$ th eigenpair of (12). Introduce the mesh  $x_n = nh$ ,  $n = 0, 1, \dots, N$ ,  $h = 1/N$ , and consider the standard second-order finite difference (FDM) discretization of (12):

$$-\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = \lambda u_n, \quad (13) \\ n = 1, \dots, N - 1, \quad u_0 = u_N = 0.$$

The corresponding approximate eigenpairs are given by

$$(\lambda_m^h, U_m^h) = \left( 4N^2 \sin^2 \frac{\pi m}{2N}, \begin{bmatrix} \sin \frac{\pi m}{N} \\ \sin \frac{\pi 2m}{N} \\ \dots \\ \sin \frac{\pi(N-1)m}{N} \end{bmatrix} \right), \\ m = 1, \dots, N - 1.$$

We also solved (12) using the MQ discretization for several values of the number  $K$  of internal nodes. Denote by  $(\lambda_m^{MQ}, U_m^{MQ})$ ,  $m = 1, \dots, K$  the corresponding approximate eigenpairs.

The results of our computations are summarized in Table 1. We use the notation  $\varepsilon_\lambda^{MQ}$ ,  $\varepsilon_\lambda^h$  for

the relative errors in  $\lambda_m^{MQ}$ ,  $\lambda_m^h$ , respectively, and the notation  $\varepsilon_U^{MQ}$  for the  $L_\infty$ -norm error in  $U_m^{MQ}$ . For each MQ solution we provide a comparison with the FDM solution that has a sufficient number of nodes to give the same accuracy for  $\lambda_1$  as the MQ method. In Part (a) of the table we use the uniform node distribution for the MQ method. Part (b) of the table shows that the accuracy of the MQ method can be significantly improved by adapting the node distribution: we moved only two nodes adjacent to boundary to reduce their distance from the boundary to  $h_1 = h/4$  (while the remaining nodes are distributed uniformly).

One can see that the MQ method can give a highly accurate solution with a small number of unknowns, 10–100 times smaller than the number of unknowns in the FDM for the same accuracy.

#### 5. Numerical Experiments for 1D and 2D Elliptic PDEs

We present several examples of continuation of solutions to systems of nonlinear 1D and 2D elliptic PDEs. Each problem is discretized by the MQ method described in Sec. 3. We then perform continuation of the resulting system of algebraic equations (11) with AUTO97. The principal goal of our examples is to assess the accuracy of the detection of bifurcation points. We compare our results with some published results and, in one case, the results of our computations with an example in AUTO97 and CONTENT. We will use throughout the notation  $K$  for the number of unknowns in a particular method. For our MQ method  $K = n \times N$ , where  $n$  is the dimension of the system and  $N$  is the number of internal nodes. We denote by MQ(u) and MQ(nu) our MQ method with the uniform and nonuniform node distributions, respectively.

**Example 1.** 1D Gelfand–Bratu equation. This is a scalar problem

$$u'' + \lambda e^u = 0, \quad \text{in } \Omega = (0, 1), \\ u(0) = u(1) = 0, \quad (14)$$

that appears in combustion theory and is used as the demo example **exp** in AUTO97 [Doedel *et al.*, 1997] (fifth-order adaptive orthogonal spline collocation method) and demo example **brg** in CONTENT [Kuznetsov & Levitin, 1995–1997] (third-order adaptive finite difference method). There is

Table 1. A 1D eigenvalue problem: comparison of results for eigenvalues, results for eigenfunctions.

(a) The MQ method with a uniform node distribution for  $K = 5, 7$  and  $9$ .

$m$	$\lambda_m$ (Exact)	$\lambda_m^{MQ}, K = 5$	Rel. Err. $\epsilon_\lambda^{MQ}$	Rel. Err. $\epsilon_U^{MQ}$	Rel. Err. $\epsilon_\lambda^h, K = 47$
1	9.86961	9.86596	$3.7 \times 10^{-4}$	$3.7 \times 10^{-4}$	$3.7 \times 10^{-4}$
2	39.4784	39.6492	$4.3 \times 10^{-3}$	$5.2 \times 10^{-3}$	$1.5 \times 10^{-3}$

  

$m$	$\lambda_m$ (Exact)	$\lambda_m^{MQ}, K = 7$	Rel. Err. $\epsilon_\lambda^{MQ}$	Rel. Err. $\epsilon_U^{MQ}$	Rel. Err. $\epsilon_\lambda^h, K = 76$
1	9.86961	9.86821	$1.4 \times 10^{-4}$	$9.9 \times 10^{-5}$	$1.4 \times 10^{-4}$
2	39.4784	39.4738	$1.2 \times 10^{-4}$	$1.8 \times 10^{-4}$	$5.7 \times 10^{-4}$
3	88.8264	89.3648	$6.0 \times 10^{-3}$	$1.1 \times 10^{-2}$	$1.3 \times 10^{-3}$

  

$m$	$\lambda_m$ (Exact)	$\lambda_m^{MQ}, K = 9$	Rel. Err. $\epsilon_\lambda^{MQ}$	Rel. Err. $\epsilon_U^{MQ}$	Rel. Err. $\epsilon_\lambda^h, K = 117$
1	9.86961	9.86901	$6.0 \times 10^{-5}$	$5.0 \times 10^{-5}$	$6.0 \times 10^{-5}$
2	39.4784	39.4846	$1.6 \times 10^{-4}$	$2.1 \times 10^{-4}$	$2.4 \times 10^{-4}$
3	88.8264	89.1667	$3.8 \times 10^{-3}$	$7.3 \times 10^{-3}$	$5.4 \times 10^{-4}$
4	157.913	159.689	$1.1 \times 10^{-2}$	$2.5 \times 10^{-2}$	$9.6 \times 10^{-4}$

(b) The MQ method with nonuniform node distribution for  $K = 7$  and  $9$ .

$m$	$\lambda_m$ (Exact)	$\lambda_m^{MQ}, K = 7$	Rel. Err. $\epsilon_\lambda^{MQ}$	Rel. Err. $\epsilon_U^{MQ}$	Rel. Err. $\epsilon_\lambda^h, K = 3477$
1	9.86961	9.86961	$6.8 \times 10^{-8}$	$3.0 \times 10^{-6}$	$6.8 \times 10^{-8}$
2	39.4784	39.4782	$3.2 \times 10^{-6}$	$3.0 \times 10^{-4}$	$2.7 \times 10^{-7}$
3	88.8264	88.8139	$1.4 \times 10^{-4}$	$6.5 \times 10^{-4}$	$6.1 \times 10^{-7}$

  

$m$	$\lambda_m$ (Exact)	$\lambda_m^{MQ}, K = 9$	Rel. Err. $\epsilon_\lambda^{MQ}$	Rel. Err. $\epsilon_U^{MQ}$	Rel. Err. $\epsilon_\lambda^h, K = 950$
1	9.86961	9.86960	$9.1 \times 10^{-7}$	$2.3 \times 10^{-6}$	$9.1 \times 10^{-7}$
2	39.4784	39.4783	$1.4 \times 10^{-6}$	$2.0 \times 10^{-5}$	$3.6 \times 10^{-6}$
3	88.8264	88.8241	$2.6 \times 10^{-5}$	$1.8 \times 10^{-4}$	$8.2 \times 10^{-6}$
4	157.913	157.882	$1.9 \times 10^{-4}$	$1.8 \times 10^{-3}$	$1.5 \times 10^{-5}$

a limit (fold) point on the solution curve. We take the value of  $\lambda$  at the limit point found from `demo` ( $K \geq 50$ ) as exact. Table 2 shows the comparison between numerical results in [Davidson, 1997], our numerical results and our experiments with CONTENT.

**Example 2.** 1D Brusselator problem, a well-known model system for autocatalytic chemical reactions with diffusion:

$$\begin{aligned} \frac{d_1}{l^2} u'' - (b+1)u + u^2 v + a &= 0, \\ \frac{d_2}{l^2} v'' + bu - u^2 v &= 0, \quad \text{in } \Omega = (0, 1), \\ u(0) = u(1) = a, \quad v(0) = v(1) &= \frac{b}{a}, \end{aligned} \quad (15)$$

proposed in [Lefever & Prigogine, 1968]. This problem exhibits a rich bifurcation scenario and has been

used in the literature as a standard model for bifurcation analysis, see e.g. [Schaeffer & Golubitsky, 1981; Golubitsky & Schaeffer, 1985; Dangelmayr, 1987; Chien *et al.*, 1997, Eq. (24)] and [Mei, 1997]. A stationary bifurcation occurs [Chien *et al.*, 1997, Eq. (24)] at

$$b_n = 1 + \frac{d_1}{d_2} a^2 + \frac{\pi^2 n^2}{l^2} d_1 + \frac{l^2}{\pi^2 n^2} \frac{a^2}{d_2} > 0.$$

For  $l = d_1 = 1$ ,  $d_2 = 2$ ,  $a = 4$ ,  $n = 1, 2$  this gives simple bifurcations:  $b_1 = 9 + \pi^2 + 8/\pi^2 = 19.680174$ ,  $b_2 = 9 + 4\pi^2 + 2/\pi^2 = 48.681060$ , correspondingly. For the second-order central difference method with uniform mesh of 41 mesh points ( $K = 80$  unknowns), the corresponding approximate bifurcation points were found in [Chien *et al.*, 1997, Sec. 6.1]. Table 3 shows the comparison between analytical, numerical results [Chien *et al.*,

Table 2. 1D Gelfand–Bratu equation: The limit point comparison.

(a) Results for the MQ method correspond to a uniform node distribution.

	[Doedel <i>et al.</i> , 1997], Exact	[Davidson, 1997], $K = 800$	MQ(u), $K = 5$	MQ(u), $K = 7$	MQ(u), $K = 9$
$\lambda$	3.513831	3.5137	3.512609	3.514224	3.514047
rel. error		$3.7 \times 10^{-5}$	$3.5 \times 10^{-4}$	$-1.1 \times 10^{-4}$	$-6.1 \times 10^{-5}$

(b) Results for the MQ method correspond to a nonuniform node distribution.

	[Kuznetsov & Levitin, 1995–1997], $K = 50$	[Kuznetsov & Levitin, 1995–1997], $K = 500$	MQ(nu), $K = 5$	MQ(nu), $K = 7$	MQ(nu), $K = 9$
$\lambda$	3.51145	3.51380	3.514010	3.513809	3.513828
rel. error	$6.8 \times 10^{-4}$	$8.8 \times 10^{-6}$	$-5.1 \times 10^{-5}$	$6.3 \times 10^{-6}$	$8.5 \times 10^{-7}$

Table 3. 1D Brusselator equation: Comparison for the bifurcation points.

(a) Results for the bifurcation point  $b_1$ .

	Exact	[Chien <i>et al.</i> , 1997], $K = 80$	MQ(u), $K = 10$	MQ(u), $K = 14$	MQ(u), $K = 18$
$b_1$	19.680174	19.67547	19.67366	19.67786	19.67919
rel. error		$2.4 \times 10^{-4}$	$3.3 \times 10^{-4}$	$1.2 \times 10^{-4}$	$5.0 \times 10^{-5}$

(b) Results for the bifurcation point  $b_2$ .

	Exact	[Chien <i>et al.</i> , 1997], $K = 80$	MQ(u), $K = 10$	MQ(u), $K = 14$	MQ(u), $K = 18$
$b_2$	48.681060	48.6004	48.57476	48.63168	48.65605
rel. error		$1.7 \times 10^{-3}$	$2.2 \times 10^{-3}$	$1.0 \times 10^{-3}$	$5.1 \times 10^{-4}$

1997, Sec. 6.1] and our numerical results for values of  $b_1$  and  $b_2$  at simple bifurcation points.

**Example 3.** Pattern formation in a 1D system with mixed boundary conditions [Dillon *et al.*, 1994].

$$\begin{aligned} \frac{d_1}{\omega l^2} u'' + \beta - \kappa u - uv^2 &= 0, \\ \delta \frac{d_1}{\omega l^2} v'' + \kappa u + uv^2 - v &= 0, \quad \text{in } \Omega = (0, 1) \\ \theta_1 \frac{\partial u}{\partial n} &= \rho(1 - \theta_1)(\theta_3 u^s - u), \\ \delta \theta_2 \frac{\partial u}{\partial n} &= \delta \rho(1 - \theta_2)(\theta_3 v^s - v), \quad \text{on } \partial\Omega = \{0, 1\}. \end{aligned} \tag{16}$$

Here  $\theta_i \in [0, 1]$ ,  $i = 1, 2, 3$ , are homotopy parameters. For  $d_1 = 10^{-5}$ ,  $\omega = 10^{-2}$ ,  $\delta = 0.14$ ,  $\beta = 1.0$ ,

$\kappa = 0.001$ ,  $(\theta_1, \theta_2, \theta_3) = (1, 1, 0)$  (Neumann problem). Equation (16) was discretized by the second-order central difference method with equidistant mesh of 41 mesh points ( $K = 80$  unknowns). Table 4 [Dillon *et al.*, 1994, Table 1] shows a comparison between analytic and numerical results for values of  $l$  at simple bifurcation points.

Our numerical results (MQ(nu) method) with  $K = 18$  coincide with the analytic results above.

**Example 4.** 2D Gelfand–Bratu problem

$$\begin{aligned} \Delta u + \lambda e^u &= 0, \quad \Omega = (0, 1) \times (0, 1), \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{17}$$

This problem was studied in [Schwetlick *et al.*, 1996] and [Doedel & Sharifi, to appear]. In [Schwetlick

Table 4. A 1D pattern formation problem, comparison for simple bifurcation points.

[[Dillon <i>et al.</i> , 1994], numerical]	0.047	0.080	0.093	0.159	0.140	0.238	0.186	0.317	0.232
[[Dillon <i>et al.</i> , 1994], analytic]	0.0465	0.0793	0.093	0.159	0.140	0.238	0.186	0.317	0.233
MQ(nu)	0.0465	0.0793	0.093	0.159	0.140	0.238	0.186	0.317	0.233

Table 5. 2D Bratu equation, results for the limit point.

(a) Uniform node distribution.				
[Schwetlick <i>et al.</i> , 1996],				
	$225 \leq K \leq 3025$	MQ(u), $K = 25$	MQ(u), $K = 49$	MQ(u), $K = 81$
$\lambda$	not reported	6.8359	6.8183	6.8119
rel. error		$4.1 \times 10^{-3}$	$1.5 \times 10^{-3}$	$5.6 \times 10^{-4}$
(b) nonuniform node distribution.				
[Doedel & Sharifi, to appear],				
	Exact	MQ(nu), $K = 25$	MQ(nu), $K = 49$	MQ(nu), $K = 81$
$\lambda$	6.808124423	6.793248	6.807978	6.808232
rel. error		$-2.1 \times 10^{-3}$	$-2.2 \times 10^{-5}$	$-1.6 \times 10^{-5}$

*et al.*, 1996] it was discretized with the second-order central difference method with uniform mesh and then continued using Implicit Block Elimination based on Recursive Projections. A limit point was detected for some value of  $\lambda$  (not reported in the paper), and spurious limit points were detected for  $K = 961, 1521, 2209, 3025$  and  $\lambda$  sufficiently small. In [Doedel & Sharifi, to appear] the problem was discretized with a high-order orthogonal spline collocation method with sparse Jacobian. We reproduced the bifurcation diagram in [Schwetlick *et al.*, 1996]. Table 5 gives the values of  $\lambda$  at the limit point computed by the MQ method. The exact location of the limit point is assumed to be at the value of  $\lambda$  obtained in [Doedel & Sharifi, to appear] on a  $16 \times 16$  mesh with  $4 \times 4$  collocation points. See Sec. 6 for a discussion of the operation count.

**Example 5.** 2D Brusselator problem.

$$\frac{d_1}{l^2} \Delta u - (b+1)u + u^2v + a = 0,$$

$$\frac{d_2}{l^2} \Delta v + bu - u^2v = 0, \quad \text{in } \Omega = (0, 1) \times (0, 1),$$

$$u|_{\partial\Omega} = a, \quad v|_{\partial\Omega} = \frac{b}{a}.$$

(18)

A stationary bifurcation occurs [Chien & Chen, 1998, Eq. (2.26)] for

$$b_{m,n} = 1 + \frac{d_1}{d_2} a^2 + d_1 \pi^2 \left( \frac{m^2}{l^2} + n^2 \right) + \frac{a^2}{\pi^2 d_2} \left( \frac{l^2}{m^2 + l^2 n^2} \right) > 0.$$

For  $l = d_1 = 1, d_2 = 2, a = 4, (m, n) = (1, 1), (m, n) = (2, 2)$  this gives simple bifurcations:  $b_{1,1} = 9 + 2\pi^2 + 4/\pi^2, b_{2,2} = 9 + 8\pi^2 + 1/\pi^2$ , correspondingly. For the second-order central difference method with equidistant mesh of 21 mesh points, the corresponding approximate bifurcation points are found in [Chien & Chen, 1998, Sec. 5]. Tables 6 and 7 show comparisons between analytical, numerical results [Chien & Chen, 1998, Eq. (2.26)] and our numerical results for values of  $b_{1,1}$  and  $b_{2,2}$  at simple bifurcation points.

A Hopf bifurcation occurs [Chien & Chen, 1998, Eq. (2.26)] for

$$b_{m,n} = 1 + a^2 + (d_1 + d_2) \left( \frac{m^2}{l^2} + n^2 \right) \pi^2$$

for some  $m, n$ , and  $l$  large enough. For  $l = 10, d_1 = d_2 = 1, a = 10, (m, n) = (1, 2)$ , this gives a

Table 6. 2D Brusselator equation: comparison for the bifurcation points, a uniform node distribution for MQ method.

(a) Results for the bifurcation point  $b_1$ .

		[Chien & Chen, 1998],			
	Exact	$K = 800$	MQ(u), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{1,1}$	29.144494	29.104774	29.16280	29.17050	29.16062
rel. error		$1.4 \times 10^{-3}$	$-6.3 \times 10^{-4}$	$-8.9 \times 10^{-4}$	$-5.5 \times 10^{-4}$

(b) Results for the bifurcation point  $b_2$ .

		[Chien & Chen, 1998],			
	Exact	$K = 800$	MQ(u), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{2,2}$	88.058156	87.47325	87.61578	87.86924	88.00143
rel. error		$6.6 \times 10^{-3}$	$5.0 \times 10^{-3}$	$2.1 \times 10^{-3}$	$6.4 \times 10^{-4}$

Table 7. 2D Brusselator equation: Comparison for the bifurcation points, a nonuniform node distribution for the MQ method.

(a) Results for the bifurcation point  $b_{1,1}$ .

	Exact	MQ(nu), $K = 50$	MQ(nu), $K = 72$	MQ(nu), $K = 98$
$b_{1,1}$	29.144494	29.14621	29.14726	29.14431
rel. error		$-5.9 \times 10^{-5}$	$-9.5 \times 10^{-5}$	$6.3 \times 10^{-6}$

(b) Results for the bifurcation point  $b_{2,2}$ .

	Exact	MQ(nu), $K = 50$	MQ(nu), $K = 72$	MQ(nu), $K = 98$
$b_{2,2}$	88.058156	88.15470	87.93391	88.07288
rel. error		$-1.1 \times 10^{-3}$	$1.4 \times 10^{-3}$	$-1.7 \times 10^{-4}$

Table 8. 2D Brusselator equation, results for the Hopf bifurcation point.

	Exact	MQ(u), $K = 50$	MQ(nu), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{1,2}$	180.15	181.8625	180.7880	181.0696	180.492
rel. error		$-9.5 \times 10^{-3}$	$-3.5 \times 10^{-3}$	$-5.1 \times 10^{-3}$	$-1.9 \times 10^{-3}$

Hopf bifurcation at  $b_{1,2} = 101 + 2((1/100) + 2^2)\pi^2 = 180.15$ , see Table 8.

## 6. Discussion

1. We have presented the results of our experiments with the continuation of solutions to 1D and 2D nonlinear elliptic PDEs discretized by the MQ method. We use a small number of unknowns and achieve a high accuracy for detected bifurcation points in our examples. Here are some sample results.

- (i) For the limit point in the 1D Gelfand–Bratu equation, the MQ method with nine unknowns gives the relative errors  $6.1 \times 10^{-5}$  and  $8.5 \times 10^{-7}$  for the uniform and nonuniform node distributions, respectively. The relative error in the third-order finite difference method with 500 nodes is  $8.8 \times 10^{-6}$ .
- (ii) For the two bifurcation points in the 2D Brusselator problem, the MQ method with 98 unknowns gives the relative errors  $5.5 \times 10^{-4}$ ,  $6.4 \times 10^{-4}$  for the uniform node distribution and  $6.3 \times 10^{-6}$ ,  $1.7 \times 10^{-5}$  for the nonuniform

node distribution. The corresponding relative errors in the second-order finite difference method with 800 nodes are  $1.4 \times 10^{-3}$ ,  $6.6 \times 10^{-3}$ .

- (iii) For the first eigenvalue in the eigenvalue problem for the 1D Laplace operator with nine unknowns gives the relative error  $6 \times 10^{-5}$  and  $9 \times 10^{-7}$  for the uniform and nonuniform node distributions, respectively. This is equivalent in accuracy to 117 and 950 node solutions, respectively by the second-order finite difference method.

2. As we noted in Introduction, the MQ method leads to systems with full matrices. Solving a related linear system for the number of nodes  $M \times M$  in 2D with a full  $M^2 \times M^2$  matrix by Gaussian elimination takes  $2/3M^6 + O(M^4)$  operations. By comparison, a band solver would take  $O(M^4)$  operations, and a collocation method on a square [Doedel, 1998; Doedel & Sharifi, to appear] would take  $\approx 62p^3M^3$ , where  $p$  is the number of matching points at an edge of a finite element [Doedel, 1998]. Further work is required to carefully compare the costs of solving linear systems and the corresponding eigenvalue problems arising in discretizing elliptic PDEs by the MQ method and by the finite difference, finite element, and collocation methods.

3. An increase of the number of unknowns and especially the shape parameter result in a better accuracy but also in a larger condition number of the operator  $\Gamma$  mapping the nodal values of the solution onto the expansion coefficients. This condition number is a limiting factor in our experiments. In fact, to reach high accuracy for the limit point in the 2D Gelfand–Bratu problem (e.g. the relative error  $1.6 \times 10^{-5}$  with 81 unknowns), we had to use quadruple precision. This is a temporary fix, as it considerably slows down computations. In future, we plan to implement the ideas of Kansa *et al.* [1990b] to circumvent this ill-conditioning problem.

We also found that even a simple adaptation of the nodes adjacent to the boundary can lead to a dramatic improvement of the accuracy in detecting bifurcation points. Adaptive choice of the shape parameter is another way to improve the accuracy that we plan to investigate.

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