SUFFICIENT CONDITIONS FOR CONTINUOUSLY-DEFINED INVARIANT SUBSPACES

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Abstract. When $A : [0, 1] \to \mathbb{C}^{n\times n}$ is a C^k function of a parameter s, standard theory tells us that the eigenvalues are continous functions of s and that for any isolated set of eigenvalues there is a maximal invariant subspace which is also a C^k function of s. In this paper, we describe sufficient conditions to test if two invariant subspaces $\mathcal{V}_0$ of $A(0)$ and $\mathcal{V}_1$ of $A(1)$ can be connected by a continuously-defined invariant subspace $\mathcal{V}(s)$ for $A(s)$ for all $s \in [0, 1]$.

1. Introduction. Suppose $A : [0, 1] \to \mathbb{C}^{n\times n}$ is a C^k matrix-valued function of a parameter s. By standard theory, we can write the eigenvalues of $A(s)$ as continuous functions $\lambda_1(s), \ldots, \lambda_n(s)$, counting multiplicity [14]. Suppose the spectrum $\Lambda(A(s))$ is partitioned into sets $\Lambda_1(s) = \{\lambda_1(s), \ldots, \lambda_m(s)\}$ and $\Lambda_2(s) = \{\lambda_{m+1}(s), \ldots, \lambda_n(s)\}$. Whenever $\Lambda_1(s) \cup \Lambda_2(s) = \emptyset$, there is a well-defined maximal invariant subspace $\mathcal{V}(s)$ associated with $\Lambda_1(s)$ which is a C^k function of s. In this paper, we study the problem of determining whether invariant subspaces $\mathcal{V}_0$ of $A(0)$ and $\mathcal{V}_1$ of $A(1)$ are associated with a continuous set of eigenvalues $\Lambda_1(s)$ which is disjoint from the rest of the spectrum for all s, so that a continuous invariant subspace $\mathcal{V}(s)$ of $A(s)$ exists satisfying $\mathcal{V}(0) = V_0$ and $\mathcal{V}(1) = A_1$.

The local structure of invariant subspaces as functions of matrices is generally treated in one of two ways. In some references, the invariant subspace is defined indirectly via a projector, and then studied via analytic function theory [14]. In numerical work, however, the invariant subspaces are often represented by explicit bases, and the treatment has a more algebraic flavor [16]. For example, by representing an invariant subspace as a reference space plus some orthogonal correction, one can derive an algebraic Riccati equation for the correction term, leading directly to both a constructive proof of error bounds for computed invariant subspaces [15] and to methods for refining approximate invariant subspaces [5, **?**]. In [9], Edelman and his colleagues proposed a more global approach to the analysis of linear algebra algorithms based on Grassmann manifolds and Stiefel manifolds (manifolds of subspaces and of orthonormal subspace bases, respectively); this approach has inspired several new methods for invariant subspace refinement, four of which are summarized and analyzed in [1]. One contribution of our work is to tie together the complex analytic, algebraic, and geometric structure of invariant subspaces.

In addition to methods to refine approximate subspaces of fixed matrices, several methods have been proposed to compute invariant subspaces for continuously parameter-dependent matrices. In [7], methods for computing a variety of continuous eigendecompositions for one-parameter matrix functions are described, including continuous Schur and block Schur decompositions. Govaerts, Guckenheimer, and Khibnik [13] proposed that in bifurcation analysis codes, a low-dimensional continuous invariant subspace could be computed at each point along a continuation path and used as the basis for bifurcation detection. This work inspired several papers on the CIS (Continuation of Invariant Subspaces) algorithm for bifurcation analysis [6, 8, 10, 11], which uses a predictor-corrector iteration to compute a continuous

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orthonormal subspace basis for a parameter dependent matrix. Though the CIS algorithm was originally intended for dense matrices, other authors have developed methods for invariant subspace continuation that exploit sparsity in the linear system [2, 4], and variants of the CIS algorithm have been developed based on Krylov subspace projections [3]. All these algorithms use heuristics to ensure that the subspace computed at each continuation steps is continuously connected to the subspace at the previous step. Our purpose is to provide an alternative to these heuristics.

The remainder of the paper is organized as follows. In Section 1, we recall a classical perturbation bound due to Stewart [15], and describe how it extends from fixed matrices to parameter-dependent matrices. We then describe a simpler alternate proof to the part of Stewart's theorem which is relevant to our analysis. In Section 2, we describe how to use interpolation to estimate the terms that occur in the hypotheses to Stewart's theorem, and in Section 3, we provide some numerical examples. In Section 4, we conclude the paper.

2. A basic perturbation result.

2.1. Fixed matrices. For this section we will consider fixed matrices $A \in \mathbb{C}^{n\times n}$ which are partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{C}^{m\times m}$ and $A_{22} \in \mathbb{C}^{(n-m)\times(n-m)}$. If A_{21} is zero, then we have an invariant subspace spanned by

$$X_0 = \begin{bmatrix} I_{m\times m} \\ 0_{(n-m)\times m} \end{bmatrix}. \tag{2.1}$$

When A_{21} is small, we expect X_0 to be a good approximation to an invariant subspace. At first order, the quality of this approximation is precisely defined in terms of $\operatorname{sep}(A_{11}, A_{22})$, where $\operatorname{sep}(B, C)$ is the smallest singular value of the Sylvester operator $X \mapsto XB - CX$. Specifically, we have that for small enough A_{12}, there is an invariant subspace spanned by a basis of the form

$$X = \begin{bmatrix} I_{m\times m} \\ Y \end{bmatrix} \tag{2.2}$$

where

$$\|Y\| \leq \frac{\|A_{21}\|}{\operatorname{sep}(A_{11}, A_{22})} + O(\|A_{21}\|^2).$$

We will take advantage of a similar result due to Stewart which is not asymptotic in nature.

THEOREM 2.1 ([16]). *Let $\|\cdot\|$ be any unitarily invariant norm. If*

$$\frac{1}{4}\operatorname{sep}(A_{11}, A_{22})^2 > \|A_{21}\|\|A_{12}\| \tag{2.3}$$

then there is a simple invariant subspace basis $X = \begin{bmatrix} I \\ Y \end{bmatrix}$ of A such that

$$\|Y\| \leq 2\frac{\|A_{21}\|}{\operatorname{sep}(A_{11}, A_{22})}. \tag{2.4}$$

To prove Theorem 2.1, Stewart writes an algebraic Ricatti equation for the correction Y from (2.2) and shows that a unique minimum solution to this equation can be obtained by fixed point iteration. This approach puts the basis of the invariant subspace in a primary role. An alternate approach, which we will develop now, is to put the eigenvalues associated with the invariant subspace in a primary role. We begin with the following lemma:

LEMMA 2.2. *A is nonsingular if*

$$\sigma_{\min}(A_{11})\sigma_{\min}(A_{22}) > \|A_{12}\|\|A_{21}\|. \tag{2.5}$$

Proof. The hypothesis implies A_{22} is nonsingular, so we can compute the Schur complement of A_{22} in A:

$$B = A_{11} - A_{12}A_{22}^{-1}A_{21}. \tag{2.6}$$

Taking norm inequalities, we find that

$$\sigma_{\min}(B) \geq \sigma_{\min}(A_{11}) - \frac{\|A_{12}\|\|A_{21}\|}{\sigma_{\min}(A_{22})}. \tag{2.7}$$

Therefore $\sigma_{\min}(B) > 0$, therefore B is nonsingular. □

If we apply the lemma to the matrix $A - zI$, we have the following inclusion result for the spectrum of A:

LEMMA 2.3. *Suppose $\delta_1\delta_2 > \|A_{12}\|\|A_{21}\|$ for some positive δ_1 and δ_2. Then*

$$\Lambda(A) \subset \Lambda_{\delta_1}(A_{11} \cup \Lambda_{\delta_2}(A_{22}) \tag{2.8}$$

where

$$\Lambda_{\delta_1}(A_{11}) := \{z \in \mathbb{C} : \sigma_{\min}(A - zI) < \delta_1\}$$
$$\Lambda_{\delta_2}(A_{22}) := \{z \in \mathbb{C} : \sigma_{\min}(A - zI) < \delta_s\}$$

The sets $\Lambda_{\delta_1}(A_{11})$ and $\Lambda_{\delta_2}(A_{22})$ are disjoint if $\delta_1 + \delta_2 < \operatorname{sep}(A_{11}, A_{22})$.

Proof. Suppose $z \notin \Lambda_{\delta_1}(A_{11}) \cup \Lambda_{\delta_2}(A_{22})$. Then

$$\sigma_{\min}(A_{11} - zI)\sigma_{\min}(A_{22} - zI) > \delta_1\delta_2 > \|A_{12}\|\|A_{21}\|$$

which by the lemma implies that $A - zI$ is nonsingular.

Now suppose $z \in \Lambda_{\delta_1}(A_{11}) \cup \Lambda_{\delta_2}(A_{22})$. Then there are perturbations Δ_{11} and Δ_{22} such that $\|\Delta_{11}\| < \delta_1$, $\|\Delta_{22}\| < \delta_2$ and $\gamma \in \Lambda(A_{11} + \Delta_{11}) \cup \Lambda(A_{22} + \Delta_{22})$, and so the Sylvester operator $X \mapsto X(A_{11} + \Delta_{11}) - (A_{22} + \Delta_{22})X$ is singular. This in turn would imply that $\operatorname{sep}(A_{11}, A_{22}) < \|\Delta_{11}\| + \|\Delta_{22}\| < \delta_1 + \delta_2$. Therefore the sets $\Lambda_{\delta_1}(A_{11})$ and $\Lambda_{\delta_2}(A_{22})$ are disjoint if $\delta_1 + \delta_2 < \operatorname{sep}(A_{11}, A_{22})$. □

Setting δ_1 and δ_2 to a common value of δ in the previous lemma, we have the following corollary.

COROLLARY 2.4. *For any $\delta > 0$ such that*

$$\frac{1}{4}\operatorname{sep}(A_{11}, A_{22})^2 > \delta^2 > \|A_{12}\|\|A_{21}\|, \tag{2.9}$$

we have that m eigenvalues of A belong to $\Lambda_\delta(A_{11})$ and the remaining $n - m$ eigenvalues belong to the disjoint set $\Lambda_\delta(A_{22})$.

The condition (2.3) from theorem 2.1 can thus also be derived as a sufficient condition under which we can distinguish a subset of the eigenvalues associated with A_{11} (i.e. eigenvalues in $\Lambda_\delta(A_{11})$) from the rest of the spectrum. The corresponding invariant subspace may be identified by computing the spectral projector associated with the contour $\partial\Lambda_\delta(A_{11})$.

2.2. Parameterized matrices. Now we return to the case where $A : [0,1] \to \mathbb{C}^{n\times n}$ is a C^k function of a parameter s. In this case, we have the following generalization of Stewart's theorem:

THEOREM 2.5. *Let $\|\cdot\|$ be any unitarily invariant norm. If for all $s \in [0,1]$*

$$\frac{1}{4}\operatorname{sep}(A_{11}(s), A_{22}(s))^2 > \|A_{21}(s)\|\|A_{12}(s)\| \tag{2.10}$$

then there is a simple invariant subspace $X(s) = \begin{bmatrix} I \\ Y(s) \end{bmatrix}$ of $A(s)$ such that

$$\|Y(s)\| \le 2\frac{\|A_{21}(s)\|}{\operatorname{sep}(A_{11}(s), A_{22}(s))}. \tag{2.11}$$

Proof. The condition (2.10) ensures that the fixed point iteration used in the proof of Stewart's theorem is uniformly convergent for all s. The iterates define continuous functions, which converge uniformly to the continuous limiting function $Y(s)$. For details, we refer to [3]. □

Theorem 2.5 provides one approach to obtaining sufficient conditions for the existence of an invariant subspace. An alternative approach admits a simpler proof, at the cost of information about the invariant subspace basis previously provided by $\|Y(s)\|$. Let us call A *block pseudo-triangular* provided

$$\frac{1}{4}\operatorname{sep}(A_{11}, A_{22})^2 > \|A_{12}\|\|A_{21}\|.$$

By Corollary 2.4, the spectrum of a block pseudo-triangular matrix can be partitioned unambiguously into disjoint sets of eigenvalues associated with A_{11} (which lie in $\Lambda_\delta(A_{11})$ and eigenvalues associated with A_{22} (which lie in $\Lambda_\delta(A_{22})$. As a consequence, we have the following theorem.

THEOREM 2.6. *For any given two-by-two block matrix partitioning, the block pseudo-triangular matrices form an open set in $\mathbb{C}^{n\times n}$. On each component of this set, the eigenprojector $P(A)$ for the eigenvalues associated with A_{11} is an analytic function of the elements of A.*

Proof. The openness of the set of pseudo-triangular matrices follows from continuity of the expressions $\operatorname{sep}(A_{11}, A_{22})$ and $\|A_{12}\|\|A_{21}\|$ with respect to A. Analyticity of the eigenprojector follows because $\Lambda_\delta(A_{11})$ remains separated from $\Lambda_\delta(A_{22})$ for all pseudo-triangular A. □

3. Interpolation bounds. Our basic approach in the rest of the paper will be to derive interpolation estimates that allow us to show that the condition (2.10) holds uniformly for a matrix similar to $A(s)$.

Suppose we are given bases for invariant subspaces of $A(s)$ at $s = 0$ and $s = h$. How can we check that the two end points are connected by a continuously defined invariant subspace basis on $[0, h]$? This question has practical significance for our continuation algorithm, since we would like to avoid mistaken branch-jumping behavior when two subspaces come close to each other, and we would like to detect when a continued invariant subspace ceases to be continuously defined.

Theorem **??** partially answers the question of how to check for a continuous connecting invariant subspace. But to apply the theorem, we need to bound $\kappa(\widehat{T})$ on the interval $[0, h]$. In the remainder of this section, we describe how to construct bounds which incorporate information from both $s = 0$ and $s = h$ using interpolation. Our ultimate goal is Theorem 3.6, but first we need some technical lemmas.

We first turn to the problem of bounding $\|B^{-1}\|_2$, where $B \in C^1([0,h], \mathbb{R}^{p\times p})$ is some parameterized operator on a Euclidean space. Since S is also a linear operator on a Euclidean space ($\mathbb{R}^{(n-m)\times m}$ with the Frobenius inner product), all our results apply directly to S as well. We begin by reviewing a simple result about matrix interpolation.

LEMMA 3.1. *Suppose $B \in C^1([0,h], \mathbb{R}^{p\times p})$ and B' is Lipschitz with constant M. Then*

$$B(s) = B(0) + B[0,h]s + B[0,h,s]s(s-h) \tag{3.1}$$

where $B[0,h]$ and $B[0,h,s]$ are first and second Newton divided differences and

$$\begin{aligned} \|B[0,h]\|_2 &\le \max_{\xi\in[0,h]} \|B'(\xi)\|_2 \\ \|B[0,h,s]\|_2 &\le M. \end{aligned}$$

Proof.

For any $u, v \in \mathbb{R}^p$ and any distinct $a, b \in [0,h]$, $a < b$, the mean value theorem applied to the scalar function $u^T B(s) v$ implies

$$u^T B[a,b] v = u^T B(\xi) v \tag{3.2}$$

for some $\xi \in [a,b]$. Therefore, $\|B[a,b]\|_2 \le \max_{\xi\in[a,b]} \|B(\xi)\|_2$.

Now we compute

$$\begin{aligned} u^T B[0,h,s] v &= (u^T B[0,s] v - u^T B[h,s] v)/h && (3.3)\\ &= (u^T B'(\xi_1) v - u^T B'(\xi_2) v)/h && (3.4)\\ &\le \frac{\|B'(\xi_1) - B'(\xi_2)\|_2}{h} \|u\|_2 \|v\|_2 && (3.5)\\ &\le M \|u\|_2 \|v\|_2. && (3.6) \end{aligned}$$

So $\|B[0,h,s]\|_2 \le M$.

∎

We can now show a very simple bound on the minimal singular value of B.

LEMMA 3.2. *Suppose $B \in C^1([0,h], \mathbb{R}^{p\times p})$ and B' is Lipschitz with constant M. Then*

$$\sigma_{\min}(B(s)) \ge \sigma_{\min}(B(0)) - \|B[0,h]\|_2 s - Ms(h-s) \tag{3.7}$$

Proof. By the previous lemma,

$$\|B(s) - B(0)\|_2 = \|B[0,h]s + B[0,h,s]s(s-h)\| \le \|B[0,h]\|_2 s + Ms(s-h).$$

To complete the proof, recall (e.g. from [12]) that

$$|\sigma_{\min}(B(s)) - \sigma_{\min}(B(0))| \leq \|B(s) - B(0)\|_2.$$

□

Lemma 3.2 uses only the norm of $B(s) - B(0)$; we can refine the bound by using the direction as well as the magnitude.

LEMMA 3.3. *Suppose $B \in C^1([0,h], \mathbb{R}^{p\times p})$ and B' is Lipschitz with constant M. Then*

$$\sigma_{\min}(B(s)) \geq \sigma_{\min}(B(0))(1 - \|B(0)^{-1}B[0,h]\|_2 s) - Ms(h-s) \tag{3.8}$$

Proof. Let $E(s) = B[0,h]s$. If $\|B(0)^{-1}E(s)\|_2 \geq 1$, then the lemma is trivial. Otherwise, $I + B(0)^{-1}E(s)$ is invertible, and

$$(B(0) + E(s))^{-1} = \left(I + B(0)^{-1}E(s)\right)^{-1} B(0)^{-1} \tag{3.9}$$

$$= \sum_{k=0}^{\infty} \left(-B(0)^{-1}E(s)\right)^k B(0)^{-1} \tag{3.10}$$

so

$$\left\|(B(0) + E(s))^{-1}\right\|_2 \leq \frac{\|B(0)^{-1}\|_2}{1 - \|B(0)^{-1}E(s)\|_2}. \tag{3.11}$$

Taking inverses on both sides, we have

$$\sigma_{\min}(B(0) + E(s)) \geq \sigma_{\min}(B(0))(1 - \|B(0)^{-1}E(s)\|). \tag{3.12}$$

Therefore

$$\sigma_{\min}(B(s)) = \sigma_{\min}(B(0) + B[0,s]s + B[0,h,s]s(s-h)) \tag{3.13}$$

$$\geq \sigma_{\min}(B(0) + B[0,s]s) - Ms(h-s) \tag{3.14}$$

$$\geq \sigma_{\min}(B(0))(1 - \|B(0)^{-1}B[0,h]\|_2 s) - Ms(h-s) \tag{3.15}$$

□

We now turn to the problem of bounding $\|F(Y_0)\|_F$ in **??** for a specific choice of Y_0. Suppose $\begin{bmatrix} I \\ hZ \end{bmatrix}$ is a basis for a given invariant subspace of $\widehat{T}(h)$ (see **??**); then we linearly interpolate $Y_0(s) = sZ$, so that the residual $F(Y_0)$ is zero at both $s = 0$ and $s = h$.

LEMMA 3.4. *Suppose $\widehat{T} \in C^1$ and $\widehat{T}'$ has Lipschitz constant M. Also suppose $\begin{bmatrix} I \\ hZ \end{bmatrix}$ spans an invariant subspace of $\widehat{T}(h)$, and define*

$$G(s) := \widehat{T}_{22}[0,h]Z - Z\widehat{T}_{11}[0,h] - Z\left(\widehat{T}_{12}(0) + (s+h)\widehat{T}_{12}[0,h]\right)Z. \tag{3.16}$$

Then for $Y_0(s) = sZ$, and for any $s \in [0,h]$,

$$\|F(Y_0)\|_F \leq \frac{h^2}{2}\left\{\max(\|G(0)\|_F, \|G(h)\|_F) + \sqrt{m}M(1 + h\|Z\|_2)^2\right\} \tag{3.17}$$

Proof.

We write $F(Y_0(s))$ as the product

$$F(Y_0(s)) = \begin{bmatrix} -Y_0(s) & I \end{bmatrix} \widehat{T}(s) \begin{bmatrix} I \\ Y_0(s) \end{bmatrix}. = \begin{bmatrix} -sZ & I \end{bmatrix} \widehat{T}(s) \begin{bmatrix} I \\ sZ \end{bmatrix}. \tag{3.18}$$

Using the Newton form of the interpolant,

$$\widehat{T}(s) = \widehat{T}(0) + \widehat{T}[0,h]s + \widehat{T}[0,h,s]s(s-h); \tag{3.19}$$

we can therefore write $F(Y_0(s))$ as

$$F(Y_0(s)) = F_1(Y_0(s)) + F_2(Y_0(s)) \tag{3.20}$$

$$F_1(Y_0(s)) = \begin{bmatrix} -sZ & I \end{bmatrix} \left(\widehat{T}(0) + \widehat{T}[0,h]s\right) \begin{bmatrix} I \\ sZ \end{bmatrix} \tag{3.21}$$

$$F_2(Y_0(s)) = \begin{bmatrix} -sZ & I \end{bmatrix} \left(\widehat{T}[0,h,s]s(s-h)\right) \begin{bmatrix} I \\ sZ \end{bmatrix}. \tag{3.22}$$

We now bound the norms of $F_1(Y_0(s))$ and $F_2(Y_0(s))$ independently.

To bound $F_1(Y_0(s))$, we expand and collect terms at each order in s:

$$F_1(Y_0(s)) = E_{21}(0) \tag{3.23}$$

$$+s\left(\widehat{T}_{22}(0)Z - Z\widehat{T}_{11}(0) + E_{21}[0,h]\right)$$

$$+s^2\left(\widehat{T}_{22}[0,h]Z - Z\widehat{T}_{11}[0,h] - Z\widehat{T}_{12}(0)Z\right)$$

$$+s^3\left(-Z\widehat{T}_{12}[0,h]Z\right) \tag{3.24}$$

Since $F(Y_0(s))|_{s=0} = 0$, we know $E_{21}(0) = 0$. Similarly, since $F(Y_0(s))|_{s=h} = 0$, we know

$$\begin{aligned} &\widehat{T}_{22}(0)Z - Z\widehat{T}_{11}(0) + E_{21}[0,h] \\ &= -h\left(\widehat{T}_{22}[0,h]Z - Z\widehat{T}_{11}[0,h] - Z\widehat{T}_{12}(0)Z\right) \\ &\quad -h^2\left(-Z\widehat{T}_{12}[0,h]Z\right). \end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.24), we have

$$\begin{aligned} F_1(Y_0(s)) = (s^2 - sh)&\left(\widehat{T}_{22}[0,h]Z - Z\widehat{T}_{11}[0,h] - Z\widehat{T}_{12}(0)Z\right) + \\ &(s^3 - sh^2)\left(-Z\widehat{T}_{12}[0,h]Z\right). \end{aligned} \tag{3.26}$$

Factoring out $s(s-h)$ from both terms, we have

$$F_1(Y_0(s)) = s(s-h)G(s). \tag{3.27}$$

Note that $G(s)$ is linear, so by convexity of norms,

$$\|G(s)\|_F \leq \max\left(\|G(0)\|_F, \|G(h)\|_F\right) \text{ for } s \in [0,h]. \tag{3.28}$$

Therefore

$$\|F_1(Y_0(s))\|_F \leq \frac{h^2}{2} \max\left(\|G(0)\|_F, \|G(h)\|_F\right) \text{ for } s \in [0,h]. \tag{3.29}$$

We use a cruder bound for $F_2(Y_0(s))$. Since $F_2(Y_0(s)) \in \mathbb{R}^{(n-m)\times m}$, $\|F_2(Y_0(s))\|_F \leq \sqrt{m}\|F_2(Y_0(s))\|_2$. Both $\begin{bmatrix}-sZ & I\end{bmatrix}$ and $\begin{bmatrix} I \\ hZ \end{bmatrix}$ are bounded in 2-norm by $1 + h\|Z\|_2$; and by 3.1, $\|\widehat{T}[0,h,s]\| \leq M$. Therefore

$$\|F_2(Y_0(s))\|_2 \leq \left\|\begin{bmatrix}-sZ & I\end{bmatrix}\right\|_2 \left\|\widehat{T}[0,h,s]\right\|_2 \left\|\begin{bmatrix} I \\ sZ \end{bmatrix}\right\|_2 s(s-h) \tag{3.30}$$

$$\leq \frac{h^2}{2} M(1 + h\|Z\|_2)^2. \tag{3.31}$$

Substituting the above bounds into $\|F(Y_0(s))\|_F \leq \|F_1(Y_0(s))\|_F + \|F_2(Y_0(s))\|_F$ concludes the proof.
□

Now we bound $\|\widehat{T}_{12}(s)\|_2$ on $[0,h]$.

LEMMA 3.5. *Suppose $\widehat{T} \in C^1$ and $\widehat{T}'$ has Lipschitz constant M. Then for $s \in [0,h]$,*

$$\|\widehat{T}_{12}(s)\|_2 \leq \max\left(\|\widehat{T}_{12}(0)\|_2, \|\widehat{T}_{12}(h)\|_2\right) + \frac{1}{2}Ms(h-s) \tag{3.32}$$

Proof. By Lemma 3.1,

$$\|T_{12}(s)\|_2 = \|T_{12}(0) + T_{12}[0,h]s + T_{12}[0,h,s]s(s-h)\|_2 \tag{3.33}$$

$$\leq \|T_{12}(0) + T_{12}[0,h]s\|_2 + Ms(h-s), \tag{3.34}$$

and because norms are convex functions,

$$\|T_{12}(0) + T_{12}[0,h]s\|_2 \leq \max\left(\|T_{12}(0)\|_2, \|T_{12}(h)\|_2\right). \tag{3.35}$$

□

Putting together the preceding bounds, we have the following theorem.

THEOREM 3.6. *Suppose $\widehat{T}(s)$ is C^2 and $\widehat{T}'$ is Lipschitz with constant M. Suppose $\begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\begin{bmatrix} I \\ hZ \end{bmatrix}$ span invariant subspaces at 0 and h respectively. Let S be defined as in (**??**). Then if*

$$\sigma_{\min}(\mathsf{S}(0))(1 - h\|\mathsf{S}(0)^{-1}\mathsf{S}[0,h]\|_2) - \frac{1}{2}Mh^2 > 0 \tag{3.36}$$

the operator S is invertible for all $s \in [0,h]$. Further, the constants α and β defined *in (??) and (??) are bounded for all $s \in [0, h]$ by*

$$\alpha \le \frac{h^2}{2} \frac{\max\left(\|G(0)\|_F, \|G(h)\|_F\right) + \sqrt{m}M(1 + h\|Z\|_2)^2}{\sigma_{\min}(\mathsf{S}(0))(1 - h\|\mathsf{S}(0)^{-1}\mathsf{S}[0,h]\|_2) - \frac{1}{2}Mh^2} \tag{3.37}$$

$$= \frac{h^2}{2} \frac{\max\left(\|G(0)\|_F, \|G(h)\|_F\right) + \sqrt{m}M}{\sigma_{\min}(\mathsf{S}(0))} + O(h^3) \tag{3.38}$$

$$\beta \le \frac{\max\left(\|\widehat{T}_{12}(0)\|_2, \|\widehat{T}_{12}(h)\|_2\right) + \frac{1}{2}Mh^2}{\sigma_{\min}(\mathsf{S}(0))(1 - h\|\mathsf{S}(0)^{-1}\mathsf{S}[0,h]\|_2) - \frac{1}{2}Mh^2} \tag{3.39}$$

$$= \frac{\max\left(\|\widehat{T}_{12}(0)\|_2, \|\widehat{T}_{12}(h)\|_2\right)}{\sigma_{\min}(\mathsf{S}(0))} + O(h) \tag{3.40}$$

where

$$G(s) = \widehat{T}_{22}[0,h]Z - Z\widehat{T}_{11}[0,h] - Z\left(\widehat{T}_{12}(0) + (s+h)\widehat{T}_{12}[0,h]\right)Z.$$

*Therefore, by Theorem **??**, if the resulting upper bound on $4\alpha\beta$ is bounded below one, there is a continuous connecting invariant subspace between $\begin{bmatrix} I \\ 0 \end{bmatrix}$ at $s = 0$ and $\begin{bmatrix} I \\ hZ \end{bmatrix}$ at $s = h$.*

Dropping higher-order terms, we have

$$\alpha \le \frac{h^2}{2} \frac{\max\left(\|G(0)\|_F, \|G(h)\|_F\right) + \sqrt{m}M}{\sigma_{\min}(\mathsf{S}(0))} + O(h^3) \tag{3.41}$$

$$\beta \le \frac{\max\left(\|\widehat{T}_{12}(0)\|_2, \|\widehat{T}_{12}(h)\|_2\right)}{\sigma_{\min}(\mathsf{S}(0))} + O(h) \tag{3.42}$$

Besides $\operatorname{sep}(\widehat{T}_{11}(0), \widehat{T}_{22}(0)) = \sigma_{\min}(\mathsf{S}(0))$ and $\|\mathsf{S}(0)^{-1}\mathsf{S}[0,h]\|_2$, the quantities in the bounds of the above theorem are cheap and simple to compute.

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